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Dimensional reduction from entanglement in Minkowski space

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Abstract: Using a quantum field theoretic setting, we present evidence for dimensional reduction of any sub-volume of Minkowksi space. First, we show that correlation functions of a class of operators restricted to a sub-volume of D-dimensional Minkowski space scale as its surface area. A simple example of such area scaling is provided by the energy fluctuations of a free massless quantum field in its vacuum state. This is reminiscent of area scaling of entanglement entropy but applies to quantum expectation values in a pure state, rather than to statistical averages over a mixed state. We then show, in a specific case, that fluctuations in the bulk have a lower-dimensional representation in terms of a boundary theory at high temperature.

Keywords: Renormalization Regularization and Renormalons, Black Holes, Statistical Methods.

Contents

1. Introduction

Physics up to energy scales of about one TeV is very well described in terms of quantum field theory which uses, roughly, one quantum mechanical degree of freedom (DOF) for each point in space. This seems to imply that the maximal entropy $S(V)$, or the dimensionality of phase space, of a quantum system in a sub-volume V is proportional to V. The Bekenstein entropy bound [1], that can be applied to any sub-volume V in which gravity is not dominant tells us that $S(V)$ has to be less than or approximately equal to the boundary area A of V in Planck units . An interpretation of this, relying on arguments involving physics of black holes, was proposed by 't Hooft and by Susskind [2] who suggested that the number of independent quantum DOF contained in a given spatial sub-volume V is bounded by the surface area of the region measured in Planck units. The holographic principle (see [3] for a recent review) postulates an extreme reduction in the complexity of physical systems. It is widely believed that quantum gravity has to be formulated as a holographic theory, and it is implicitly assumed that gravity is somehow responsible for this massive reduction of the number of DOF. This point of view has received strong support from the AdS/CFT duality (see [4] for a review), which defines quantum gravity non-perturbatively in a certain class of space-times and explicitly exposes only the physical variables admitted by holography.

Another route leading to area dependent entropy originated in evidence of dimensional reduction due to quantum entanglement, mainly from the calculation of the von Neumann entropy of sub-systems. Srednicki [5] (and previously Bombelli et. al. [6]) considered the von Neumann entropy of quantum fields in a state defined by taking the vacuum and tracing over the DOF external to a spherical sub-volume of Minkowski space. They discovered that this "entropy of entanglement" was proportional to the boundary area of the sub-volume. It is possible to show that the entropy obtained by tracing over the DOF outside of a sub-volume of any shape is equal to the one obtained by tracing over the DOF inside this sub-volume [5] but an explicit numerical calculation was needed to show that it is linear in the area for a spherical sub-volume. Following this line of thought, some other geometries were considered, in particular, half of Minkowski space in various dimensionalities (see for example, [7]–[11]). Dimensional reduction in this case seems to be quite different from that in AdS, and its relationship to dimensional reduction due to gravity was never clarified (see however [12, 13, 14]).

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Following a calculation in [15], we have discovered that in the vacuum state, energy fluctuations of a free massless scalar field in a sub-volume of Minkowski space are proportional to the boundary area of this sub-volume. Energy fluctuations of quantum fields in their vacuum state in the whole of Minkowski space vanish, of course, because the vacuum is an eigenstate of the hamiltonian. But, energy fluctuations in a sub-volume of Minkowski space do not vanish.[1] Not only do they not vanish, but they are divergent. Once a high-momentum cutoff is introduced to regularize the divergence, they depend on a positive power of the cutoff. The power is determined by dimensional analysis such that the result has the correct dimensionality. We will discuss the significance of the divergences in detail after presenting a concrete example.

The origin of such energy fluctuations is, in some sense, similar to the origin of entanglement entropy discussed above except that they do not involve tracing over DOF, and are calculated as quantum expectation values in a pure state, rather than as statistical averages over a mixed state. Area scaling is not specific to energy fluctuations of free massless fields. We show that it is valid for correlation functions of a class of operators and persists for interacting field theories. Here, also, the correlation functions diverge, with a power dependence on the UV cutoff. The power being determined by the dimensionality of the operator. In a specific case we show that the area scaling of correlation functions results from a boundary field theory at high temperature.

Let us start with a simple example: energy fluctuations of a free massless quantum field theory in its vacuum state in a sub-volume V of Minkowski space. We define the (normal-ordered) energy operator for this sub-volume $E^V = \int_V :\mathcal{H}(\vec{x}): d^d x$, where $:\mathcal{H}(\vec{x}):$ is the normal-ordered zero-zero component of the energy-stress tensor, i.e., the hamiltonian density. For a free massless field theory in d spatial dimensions it is possible to express E^V in terms of canonical creation and annihilation operators $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$:

$$E^V = \frac{1}{4}\frac{1}{(2\pi)^{2d}}\int_V d^d x \int \frac{d^d p}{\sqrt{p}}\frac{d^d q}{\sqrt{q}} e^{\imath(\vec{p}+\vec{q})\cdot\vec{x}}\left\{2(pq - \vec{p}\cdot\vec{q})a^\dagger_{\vec{p}}a_{\vec{q}} - (\vec{p}\cdot\vec{q}+pq)\left(a_{\vec{p}}a_{\vec{q}} + a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}\right)\right\}.$$

The vacuum expectation value of E^V vanishes: $\langle 0|E^V|0\rangle = 0$. However, the energy fluctuations do not:

$$\langle 0|(E^V)^2|0\rangle = \frac{1}{8}\frac{1}{(2\pi)^{2d}}\int_V d^d y_1 \int_V d^d y_2 \int d^d p \int d^d q e^{\imath(\vec{p}+\vec{q})\cdot(\vec{y}_1-\vec{y}_2)}\left(\frac{\vec{p}\cdot\vec{q}}{\sqrt{pq}} + \sqrt{pq}\right)^2. \quad (1.1)$$

After performing the integration over the momenta, the integrand is a function of the variable $|\vec{y}_1 - \vec{y}_2|$. Rewriting eq.(1.1) as $\langle (E^V)^2\rangle = \int_0^\infty F(y)\mathcal{D}_V(y)dy$, where

$$F(y) = \frac{1}{8}\frac{1}{(2\pi)^{2d}}\int \left(pq + 2\vec{p}\cdot\vec{q} + \frac{(\vec{p}\cdot\vec{q})^2}{pq}\right) e^{-\imath(\vec{p}+\vec{q})\cdot\vec{y}} d^d p\, d^d q\,,$$

and

$$\mathcal{D}_V(y) = \int_V d^d y_1 \int_V d^d y_2\, \delta^{(d)}(y - |\vec{y}_1 - \vec{y}_2|)\,, \quad (1.2)$$

the integral is separated into a geometric factor, $\mathcal{D}_V(y)$, which depends only on the shape of the sub-volume V, and another factor $F(y)$.

[1] As the sub-volume in question is of the order of the volume of Minkowski space, the energy fluctuations rapidly decrease to zero.

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Evaluating $\langle 0|(E^V)^2|0\rangle$ we find, remarkably, that it is proportional to the boundary area $S(V)$ of V for any volume whose boundary is regular. It is possible to compute the energy fluctuations analytically with an exponential high momentum (UV) cutoff Λ:

$$\langle 0|(E^V)^2|0\rangle = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(2+\frac{d}{2}\right)} \frac{1}{2^{d+4}\pi^{\frac{d}{2}+1}} \Lambda^{d+1} S(V)\,.$$

(See [16] for details.) These fluctuations can also be evaluated numerically using different cutoff schemes [17]. One can prove that the area dependence is a robust result, while the numerical coefficients depend on the details of the cutoff procedure. This area scaling behavior may have been inferred by noting that if we denote the complement of V by $\widehat{V}$ and that $S(V) = S(\widehat{V})$ then $\langle 0|(E^{\widehat{V}})^2|0\rangle = \langle 0|(E^V)^2|0\rangle$.

The surprising result that energy fluctuations in a sub-volume of Minkowski space are proportional to the boundary area of the sub-volume is not accidental. Rather, it seems to be a general characteristic property of sub-systems. We wish to understand this fact, and examine how general it is, which we do in the rest of the paper.

2. Operators in a finite sub-volume and their correlation functions

As a generalization of the above, we wish to examine expectation values of operators of the form $\langle O_i^V O_j^V\rangle$ defined as follows. Let us consider a finite d dimensional space-like domain of volume V and linear size R, which we think of as being part of a $d = D-1$ dimensional Minkowski space. A quantum field theory is defined in the whole space. Generically, it is an interacting field theory, which comes equipped with a high momentum (UV) cutoff Λ, and some regularization procedure. A low momentum (IR) cutoff is implicitly assumed, but we will not discuss any of its detailed properties. The sub-volume is "macroscopic" in the sense that $R\Lambda \gg 1$. The boundary of V is an "imaginary" boundary, since we do not impose any boundary conditions, or restrictions on the fields on it.

We will be interested in a set of (possibly composite) operators O_i which can be expressed as an integral over a density $\mathcal{O}_i$: $O_i = \int d^d x \mathcal{O}_i(\vec{x})$. For this set we may define the operators $O_i^V = \int_V d^d x \mathcal{O}_i(\vec{x})$. We will discuss for concreteness the case in which the large space is in the vacuum state $|0\rangle$. Our results can be easily generalized to states other than the vacuum. We note that the vacuum will not be an eigenstate of O_i^V.

We have already considered energy fluctuations in a given sub-volume V. In general, in addition to fluctuations of operators, namely, two point functions of the same operator, we may look at two point functions of different operators. Such correlation functions also scale linearly with the boundary area of the sub-volume V. To show this, we would like to evaluate the correlation function $\langle 0|O_i^V O_j^V|0\rangle = \int_V d^d y_1 \int_V d^d y_2\ F_{ij}\left(|\vec{y}_1 - \vec{y}_2|\right)$. Following [15], we evaluate all the integrals except for the $y = |\vec{y}_1 - \vec{y}_2|$ integral,

$$\int_V d^d y_1 \int_V d^d y_2\ F_{ij}\left(|\vec{y}_1 - \vec{y}_2|\right) = \int_0^\infty dy\, \mathcal{D}_V(y)\, F_{ij}(y)\,.$$

The geometric factor $\mathcal{D}_V(y)$ can be evaluated explicitly. By a careful geometric analysis, one can show that $D_V(y) = G_V V y^{d-1} + G_S S(V) y^d + \mathcal{O}(y^{d+1})$, G_V and G_S being

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constants, and $S(V)$ the surface area of V [16]. We will need to use $\mathcal{D}_V(y) \sim y^{d-1}$ for $y \sim 0$. This property may be shown by considering eq. (1.2). The limit $y \sim 0$ is approached when both vectors $\vec{y}_1$ and $\vec{y}_2$ are almost equal. Fixing one of them, say $\vec{y}_1$, at some arbitrary value makes the integral over $\vec{y}_2$ an unrestricted d-dimensional integral, which in spherical coordinates yields the above result. Defining $\mathcal{D}_V(y) = y^{d-1} R^d \widetilde{\mathcal{D}}_V\left(\frac{y}{R}\right)$, which captures the function's behavior near $y \sim 0$, and its dimensionality (recall that R is the linear scale of V). We note that for a generic sub-volume $\widetilde{\mathcal{D}}'_V(0) \neq 0$. We would like to emphasize that in general R^d will be replaced by a volume factor and R^{d-1} will be replace by the surface area of the boundary as discussed exhaustively in [16]. We will continue the analysis using the expressions R^d, and R^{d-1} as the general case is somewhat encumbering.

The short distance/large momentum behavior of correlation functions is determined by the full mass dimensionalities δ_i, δ_j (including anomalous dimensions induced by interactions) of the operator densities $\mathcal{O}_i$ and $\mathcal{O}_j$. At short distances $F_{ij}(|\vec{x}|) \sim 1/|\vec{x}|^{\delta_i+\delta_j}$, equivalently for large momentum, $\widetilde{F}_{ij}(|\vec{q}|) \sim |\vec{q}|^{\delta_i+\delta_j-d}$, where we have defined $\widetilde{F}_{ij}(|\vec{q}|) = \int d^d x\, e^{i\vec{q}\cdot\vec{x}} F_{ij}(|\vec{x}|)$. Here we have used the leading short-distance behavior. In general, $\widetilde{F}_{ij}(|\vec{q}|)$ will be a function of several powers of q, which may be fractional or perhaps multiplied by logarithms. We will consider operators whose correlation functions are sharply peaked at short distances, and decay at large distances. The short distance/large momentum behavior of correlation functions is in many cases singular before a UV cutoff is introduced and needs to be regularized. We do this by cutting off the momentum integrals that appear in the expressions for the two-point functions. The significance of these divergences will be discussed in detail later. The details of the regularization procedure and the exact nature of the UV cutoff are not particularly important since our main observation is the area scaling nature of the fluctuations.

So after all is said and done, we need to evaluate the following integral,

$$\int_0^\infty dy\, \mathcal{D}_V(y)\, F_{ij}(y) = R^d \int_0^\infty dy\, y^{d-1}\, \widetilde{\mathcal{D}}_V\left(\frac{y}{R}\right) \int \frac{d^d q}{(2\pi)^d}\, e^{-iqy\cos\theta_q}\, \widetilde{F}_{ij}(|\vec{q}|)\,, \tag{2.1}$$

where θ_q is the angle of the vector $\vec{q}$ with respect to some fixed axis. The last momentum integral is regularized by the UV cutoff of the theory. Without such a cutoff it is divergent in the large momentum region whenever $\delta_i + \delta_j > 0$. We choose to regulate the integral in eq. (2.1) by an additional factor that suppresses the high momentum contribution.

Integrals of the form appearing in eq. (2.1) can be evaluated analytically for $F(|\vec{q}|) \propto (q^2)^n$ with n integer and a gaussian momentum, and for $F(|\vec{q}|) \propto q^\alpha$ with α rational and an exponential momentum cutoff. We will discuss in some detail the integer case, and quote the results for the other case.

For a gaussian cutoff eq. (2.1) with $F_{ij}(q) \sim q^{2n}$, reduces to

$$I_n = R^d \int_0^\infty dy\; y^{d-1}\; \widetilde{\mathcal{D}}_V\left(\frac{y}{R}\right) \int d^d k\, e^{iky\cos\theta_k - \frac{1}{2}k^2/\Lambda^2} (k^2)^n\,. \tag{2.2}$$

Naively, this integral scales as $R^d \Lambda^{2n}$ (extensively) as the pre-factor might suggest. Now, $|\vec{k}|^2$ may be rewritten as a d-dimensional laplacian in spherical coordinates: $\frac{1}{y^{d-1}}\partial_y y^{d-1}\partial_y$.

If we now integrate by parts once, the volume term will vanish. Due to the gaussian, we may approximate $y \sim 0$, and we are left with

$$I_n = a_V \; C_d \; R^{d-1} \Lambda^{2n-1} \,, \tag{2.3}$$

where

$$C_d \sim (-1)^n \int_0^\infty dx x^{d-2} \left[\left(\frac{1}{x^{d-1}} \partial_x x^{d-1} \partial_x \right)^{n-1} e^{-\frac{1}{2}x^2} \right]$$

is a finite d- and n-dependent remnant of the momentum integral, and $a_V = (2\pi)^{d/2} \, \widetilde{\mathcal{D}}'_V(0)$. Thus we have shown that I_n scales as $R^{d-1}\Lambda^{2n-1}$. Similarly

$$I_\alpha = R^d \int_0^\infty dy \, y^{d-1} \widetilde{\mathcal{D}}_V \left(\frac{y}{R} \right) \int d^d k \, e^{i\vec{k}\cdot\vec{y} - k/\Lambda} k^\alpha \sim \Lambda^{\alpha-1} R^{d-1} \,,$$

where α is not necessarily integer.

It is possible to show that the area scaling law $\langle O_i^V O_j^V \rangle \propto \Lambda^{\delta_i+\delta_j-1} R^{d-1}$ is robust to changes in the cutoff procedure. Any cutoff function $C(k/\Lambda)$ will give similar results provided that it and all of its derivatives vanish at $k \to \infty$, and that its integral $\int d^d k C(k/\Lambda)$ is finite. It is also possible to show [16] that if the connected two point function of the operator densities $F_{ij}(y) \equiv \nabla^2 g_{ij}(y)$ satisfies (i) $g_{ij}(y)$ is short range: at large distances it decays fast enough $|g'(y)| < 1/y^{a+1}$, with $a \geq d-1$, and (ii) $g_{ij}(y)$ is not too singular at short distances: for small y, $|g(y)| < 1/y^{a+1}$ with $a < d-2$, then the connected two point function $\langle O_i^{V_1} O_j^{V_2} \rangle_C$ is proportional to the area of the common boundary of the two regions V_1 and V_2. We would like to emphasize that our results are valid for any function that obeys the above conditions, and not only by correlation functions that have strictly power law dependence on the distance.

The divergence of the correlation functions cannot be renormalized in the standard way by adding local counter terms to the action. This is because the coefficient of the divergence depends on the area, and therefore on the geometry of the imaginary sub-volume in which they are calculated. Additionally, they appear at the level of free field theory which is usually not renormalized. This suggests that the divergence of the correlation functions is related to a physical UV cutoff of the theory given by, say, a Planck scale lattice or some other UV complete theory. If a physical UV completion of the field theory exists, then in the framework of that UV completion the divergences become indeed finite and the cutoff has a physical meaning.

3. Lower-dimensional representation of correlation functions

The fact that correlation functions of operators scale as the boundary area of the sub-volume rather than as the volume (as expected), suggests that it might be possible to find a representation of them on the boundary ∂V of V (and of $\widehat{V}$). To show this we shall express $\langle 0|O_i^V O_j^V|0\rangle$ as a double derivative $\langle 0|O_i^V O_j^V|0\rangle = -\int_V d^d y_1 \int_V d^d y_2 \vec{\nabla}_1 \cdot \vec{\nabla}_2 g_{ij}(|\vec{x} - \vec{y}|)$ and use Gauss' law to express it as a boundary correlation function, $\langle 0|O_i^V O_j^V|0\rangle =$

$\oint_{\partial V} d^{d-1}z_1 \oint_{\partial V} d^{d-1}z_2 \hat{n}_1 \cdot \hat{n}_2 g_{ij}(|\vec{z}_1 - \vec{z}_2|)$. This allows us to define an equivalent correlation function in $d-1$ dimensions of some operators $\Theta_i^{\partial V}$ and $\Theta_j^{\partial V}$

$$\langle 0|O_i^V O_j^V|0\rangle \sim \langle 0|\Theta_i^{\partial V}\Theta_j^{\partial V}|0\rangle_{d-1} = \alpha_{ij}^d \int_{\partial V} d^{d-1}z_1 \int_{\partial V} d^{d-1}z_2 \ \hat{n}_1 \cdot \hat{n}_2 \langle 0|\vartheta_i(\vec{z}_1)\vartheta_j(\vec{z}_2)|0\rangle_{d-1}\,. \tag{3.1}$$

Here $\hat{n}$ is a boundary unit normal. For example, in the case that the correlation functions are pure powers $\frac{c_{ij}^d}{|\vec{y}_1 - \vec{y}_2|^{\delta_i+\delta_j}}$, then $\alpha_{ij}^d = \frac{c_{ij}^d}{(\delta_i+\delta_j-2)(d-\delta_i-\delta_j)c_{ij}^{d-1}}$. In the special case $\delta_i + \delta_j = 2$ the power becomes a logarithm, and for $\delta_i + \delta_j = d$ it becomes a delta function, so they need to be handled with special care. Note that $|\vec{z}_1 - \vec{z}_2|$, even though evaluated for $\vec{z}_1$ and $\vec{z}_2$ on the boundary, generally expresses distances in the bulk. In the case that the boundary is a plane, $|\vec{z}_1 - \vec{z}_2|$ indeed expresses distances on the boundary.

It will be very interesting to determine the consistency conditions under which a boundary representation of integrals of bulk correlation functions can be derived from an effective action, and to determine how symmetries of the sub-volume and of the bulk theory are reflected in the properties of their boundary representations. A covariant formulation, starting from a lagrangian in $D = d+1$ space-time dimensions would be very useful. At this point we would like to present evidence of a special case where the boundary theory is a theory at high-temperature, and leave the answers to the questions above open for future research.

We consider a massless, free scalar field theory and take V to be half of Minkowski space (under some assumptions it is possible to generalize this to other geometries.) We first show that the n-point (single field) functions for a $(d+1)$ dimensional free scalar field theory in the vacuum state are equal to n-point functions of a $((d-1)+1)$ dimensional free scalar field theory at high-temperature,

$$\langle 0|e^{-J\int_V \phi(\vec{x})d^dx}|0\rangle_d = \left\langle e^{-J\sqrt{\alpha_{\phi\phi}^d}\int_{\partial V}\phi(\vec{x})d^{d-1}x}\right\rangle_{d-1}^{\beta\to 0}. \tag{3.2}$$

We shall show this equality term by term,

$$\begin{aligned}\int_V d^dx_1 \ldots \int_V d^dx_{2n}\langle 0|\phi(\vec{x}_1)\cdots\phi(\vec{x}_{2n})|0\rangle_d &= \\ = (\alpha_{\phi\phi}^d)^n \int_{\partial V} d^{d-1}x_1 \cdots \int_{\partial V} d^{d-1}x_{2n}\langle\phi(\vec{x}_1)\cdots\phi(\vec{x}_{2n})\rangle_{d-1}^{\beta\to 0}\,.\end{aligned}$$

For an odd number of fields this is trivially satisfied, since both sides vanish. For an even number of free fields in any dimension,

$$\langle 0|\phi(\vec{x}_1)\cdots\phi(\vec{x}_{2n})|0\rangle = \sum_{\substack{\text{all}\\ \text{perm.}}} \langle 0|\phi(\vec{x}_{i_1})\phi(\vec{x}_{i_2})|0\rangle \cdots \langle 0|\phi(\vec{x}_{i_{2n-1}})\phi(\vec{x}_{i_{2n}})|0\rangle\,.$$

So to prove eq. (3.2) we only need to show that

$$\int_V d^dx_1 \int_V d^dx_2 \langle 0|\phi(\vec{x}_1)\phi(\vec{x}_2)|0\rangle_d = \alpha_{\phi\phi}^d \int_{\partial V} d^{d-1}x_1 \int_{\partial V} d^{d-1}x_2 \langle\phi(\vec{x}_1)\phi(\vec{x}_2)\rangle_{d-1}^{\beta\to 0}\,.$$

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The mass dimension of a scalar field in $d+1$ dimensions is $\delta = \frac{d-1}{2}$, so

$$\begin{aligned}\int_V d^d x_1 \int_V d^d x_2 \langle \phi(\vec{x}_1)\phi(\vec{x}_2)\rangle_d &= \alpha^d_{\phi\phi} \int_{\partial V} d^{d-1} x_{1_\perp} \int_{\partial V} d^{d-1} x_{2_\perp} \frac{\hat{n}_1 \cdot \hat{n}_2}{|\vec{x}_1 - \vec{x}_2|^{d-3}} \\ &= \alpha^d_{\phi\phi} \int_{\partial V} d^{d-1} x_{1_\perp} \int_{\partial V} d^{d-1} x_{2_\perp} \langle \phi(\vec{x}_1)\phi(\vec{x}_2)\rangle^{\beta\to 0}_{d-1} . \qquad (3.3)\end{aligned}$$

In eq. (3.3) we have used the fact that a $D = d+1$ dimensional field theory in the limit of high temperature is equivalent to a $D-1$ dimensional field theory at zero temperature, and that $\hat{n}_1 \cdot \hat{n}_2 = 1$ when V is half of Minkowski space. Our choice of viewing the power of the integrand as that of a $(d-1)$ dimensional theory is motivated by the region of integration. One should note that this procedure is quite distinct from the well-known dimensional reduction of a high-temperature quantum field theory. Here we relate a $d+1$ dimensional theory to a $d-1$ dimensional theory, which is equivalent to a d dimensional theory at high temperature.

Similarly, such dimensional reduction may also be explicitly shown for other two point functions

$$\begin{aligned}&\int_V d^d x_1 \int_V d^d x_2 \langle 0|\nabla_1^m \phi^n(\vec{x}_1) \nabla_2^{m'} \phi^{n'}(\vec{x}_2)|0\rangle_d \cong \\ &\cong \int_{\partial V} d^{d-1} x_{1_\perp} \int_{\partial V} d^{d-1} x_{2_\perp} \langle \nabla_1^{m+n-1} \phi^n(\vec{x}_1) \nabla_2^{m'+n-1} \phi^{n'}(\vec{x}_2)\rangle^{\beta\to 0}_{d-1} ,\end{aligned}$$

where by $\nabla_1^m \nabla_2^{m'}$, we mean the scalar operator obtained by consecutive operations of ∇; in order for this to be a scalar operator we must have that $m + m'$ is even. This may also be generalized to the case of single field n-point functions with an arbitrary number of derivatives.

We are able to extend our conclusions to the most general correlation functions in the theory - products of powers of fields and their derivatives, only in the case of a large number N of fields (in which case the correlation functions reduce to products of two-point functions.) To see this explicitly and to discuss the large-N limit we define $\Phi(x_i) = \mathrm{diag}(\phi_1(x_i), \ldots, \phi_N(x_i))$. In this limit only correlations functions of the form $\int_V \cdots \int_V \langle \mathrm{Tr}\, \Phi(x_1) \ldots \mathrm{Tr}\, \Phi(x_n)\rangle d^d x_1 \ldots d^d x_n$ contribute to leading (zero) order in the $1/N$ expansion. Thus, to first order in the large-N limit, all bulk correlation functions have a boundary description.

The fact that the boundary theory seems to be in a high temperature state is reminiscent of the thermal atmosphere near black hole horizons. The local Hawking temperature increases from its asymptotic value as one gets near the horizon. In fact, there were some attempts to explain black hole entropy and its area scaling as originating from the thermal entropy of the near-horizon thermal atmosphere [19, 20, 21]. Another connection with black hole physics is the area dependence of thermodynamic quantities. The fluctuations we have described are quantum. However, if instead of quantum expectation values of operators restricted to V, we consider fluctuations in the state $\rho_V = \mathrm{Tr}_{\widehat{V}} |0\rangle\langle 0|$, we get from $\langle 0|O^V|0\rangle = \mathrm{Tr}(\rho_V O^V)$ that area scaling of quantum fluctuations leads to area dependent thermodynamics [18].

The nature of the dimensional reduction that we have discussed seems to be more restricted than the holographic correspondence in AdS/CFT: we have presented evidence that a class of operators that can be expressed as an integral over a density in the bulk can be represented by an integral over a density on the boundary. On the other hand the AdS/CFT duality conjectures a correspondence between local operators in the bulk and in the boundary. Since we have not used gravity to establish our correspondence, this raises the possibility that both gravity and quantum mechanics have distinct functionality in holography.

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